Exercise 8

Repeat Exercise 5(a) for the data $u(x,0)=0$, $\frac{\partial u}{\partial t}(x,0)=\frac{x}{(1+x^2)^2}$, $-\infty<x<\infty$.

Solution

The aim is to solve the wave equation on the whole line for all time subject to initial conditions.

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}, \quad -\infty<x<\infty,\ -\infty<t<\infty$$
$$u(x,0)=f(x)=0$$
$$\frac{\partial u}{\partial t}(x,0)=g(x)=\frac{x}{(1+x^2)^2}$$

Start with the general solution of the wave equation.

$$u(x,t)=F(x+ct)+G(x-ct)$$

Differentiate it with respect to t.

$$\frac{\partial u}{\partial t}=F'(x+ct)\cdot\frac{\partial}{\partial t}(x+ct)+G'(x-ct)\cdot\frac{\partial}{\partial t}(x-ct)=F'(x+ct)\cdot(c)+G'(x-ct)(-c)=cF'(x+ct)-cG'(x-ct)$$

Now apply the given initial conditions.

$$u(x,0)=F(x)+G(x)=f(x)$$
$$\frac{\partial u}{\partial t}(x,0)=cF'(x)-cG'(x)=g(x)$$

This is a system of two equations with two unknowns, F and G, that can be solved for. Differentiate both sides of the first equation.

$$\begin{cases} F'(x)+G'(x)=f'(x) \\ cF'(x)-cG'(x)=g(x)\end{cases}$$

Multiply both sides of the first equation by c.

$$\begin{cases} cF'(x)+cG'(x)=cf'(x) \\ cF'(x)-cG'(x)=g(x)\end{cases}$$

Adding the respective sides of these equations eliminates G and gives

$$2cF'(x)=cf'(x)+g(x).$$

Divide both sides by $2c$.

$$F'(x)=\frac{1}{2}f'(x)+\frac{1}{2c}g(x)$$

Integrate both sides with respect to x.

$$F(x)=\frac{1}{2}f(x)+\frac{1}{2c}\int^x g(s)\,ds+C_1$$

Subtracting the respective sides of these equations instead eliminates F and gives

$$2cG'(x) = cf'(x) - g(x).$$

Divide both sides by $2c$.

$$G'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x)$$

Integrate both sides with respect to x.

$$\begin{aligned} G(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int^x g(s)\,ds + C_2 \\ &= \frac{1}{2}f(x) + \frac{1}{2c}\int_x g(s)\,ds + C_2 \end{aligned}$$

Now that F and G are known, the solution to the initial value problem can be written.

$$\begin{aligned} u(x,t) &= F(x+ct) + G(x-ct) \\ &= \left[\frac{1}{2}f(x+ct) + \frac{1}{2c}\int^{x+ct} g(s)\,ds + C_1\right] + \left[\frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct} g(s)\,ds + C_2\right] \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds + C_3 \end{aligned}$$

The integration constant is set to zero to satisfy $u(x,0) = f(x)$.

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

In this exercise

$$f(x) = 0 \quad \text{and} \quad g(x) = \frac{x}{(1+x^2)^2},$$

so

$$u(x,t) = \frac{1}{2}(0+0) + \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{s}{(1+s^2)^2}\,ds.$$

Make the following substitution.

$$v = 1 + s^2$$

$$dv = 2s\,ds \quad \to \quad \frac{dv}{2} = s\,ds$$

As a result,

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{1+(x-ct)^2}^{1+(x+ct)^2} \frac{1}{v^2}\left(\frac{dv}{2}\right) \\ &= \frac{1}{4c}\int_{1+(x-ct)^2}^{1+(x+ct)^2} v^{-2}\,dv \\ &= \frac{1}{4c}\left(-\frac{1}{v}\right)\Bigg|_{1+(x-ct)^2}^{1+(x+ct)^2} \\ &= \frac{1}{4c}\left[\frac{1}{1+(x-ct)^2} - \frac{1}{1+(x+ct)^2}\right]. \end{aligned}$$

Below are plots of $u(x,t)$ versus x over $-15 < x < 15$ for $t = 0, 1, 2, 4, 6, 8$ with $c = 1$.